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A note on the coexistence of infinitely many attractors

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ABSTRACT

This note provides a simple argument useful for verifying the coexistence of infinitely many attractors in many dynamical systems with periodic components. It demonstrates that, for some systems, if one attractor exists then there would be infinitely many. This argument works for both continuous-time and discrete-time settings.

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The coexistence of finitely many or infinitely many attractors is an important feature of a complex dynamical system. The argument established in this note can be applied to many systems in the literature, in which the coexistence of infinitely many attractors has been confirmed, provided that one attractor can be found. The same argument can be used to show the coexistence of infinitely many periodic orbits or quasi-periodic orbits. This reveals an interesting phenomenon: "One plus periodicity imply infinity."

I. INTRODUCTION

The coexistence of several attractors in a dynamical system is a very special dynamical behavior, also referred to as the multistability (coexistence of finitely many attractors) or extreme multistability (coexistence of infinitely many attractors), which can be found from many dynamical systems in various fields, such as mathematics, physics, chemistry, biology, economy, and so on.[1]

There exist quite a large number of continuous-time systems with the coexistence of multiple attractors. Taking the well-known Chua's circuit as an example, many variants of the Chua's circuit have coexistence of multiple attractors, for instance, an active diode-based Chua's circuit,[2] a simplified canonical Chua's oscillator with smooth hyperbolic sine function,[3] a modified fourth-order Chua's chaotic system,[4,5] a non-ideal active voltage-controlled memristive Chua's circuit,[6] a memristor-based modified Chua's circuit,[7] and so on. There are also different classes of systems with the coexistence of infinitely many attractors.[8–10]

In many discrete-time systems, the coexistence of multiple attractors is common. For example, sinusoidal functions can be applied to generate hyperchaotic dynamics with multiple attractors.[11] A class of discrete-time systems with two sinusoidal functions having irrational frequency ratios can have multiple coexisting attractors.[12] Even a one-dimensional trigonometric chaotic system with multiple attractors can be constructed using one or two sinusoidal functions.[13]

In the study of the coexistence of multiple attractors, some special contributions of periodic components of the systems are designed. This note provides a simple argument useful for verifying the coexistence of infinitely many attractors in many dynamical systems with periodic components. It demonstrates that, for some systems, if one attractor exists then there would be infinitely many. This argument works for both continuous-time and discrete-time settings. It is also shown that, not only the appearance of multiple attractors, but also their geometric structures often display some similarity, as observed in continuous systems (such as Figs. 9 and 16 of Ref. 14 and Figs. 1–7 of Ref. 10) and discrete systems (such as Figs. 1 and 2 of Ref. 15, Fig. 6 of Ref. 16, and Figs. 11–14 of Ref. 17)

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before. This note also explains why the geometric structures of these attractors are similar.

The rest of the note is organized as follows. In Sec. II, the continuous-time systems are studied. In Sec. III, the discrete-time systems are investigated.

II. CONTINUOUS-TIME SYSTEMS

In this subsection, continuous-time systems with infinitely many attractors are studied.

For simplicity, only three-dimensional systems are discussed. The argument derived here can be easily extended to dynamical systems in $\mathbb{R}^n$ with some other types of periodic functions.

Definition 1 (Ref. 18): Let $F : \mathbf{x} \in \Omega_1 \subset \mathbb{R}^3 \to \mathbf{w} \in \Omega_2 \subset \mathbb{R}^3$ be a function. The function F is said to be a diffeomorphism if F is differentiable, one-to-one, and surjective, and its inverse is also differentiable, one-to-one, and surjective.

Consider the following coordinate transformation of $\mathbf{x} = (x_1, x_2, x_3)$:

$$\begin{cases} w_1 = f_1(x_1, x_2, x_3), \\ w_2 = f_2(x_1, x_2, x_3), \\ w_3 = f_3(x_1, x_2, x_3), \end{cases} \tag{1}$$

where f_i are differentiable, $i = 1, 2, 3$. Denote $F := (f_1, f_2, f_3)$. The Jacobian of this transformation is

$$\mathrm{Jac}(F) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{pmatrix}.$$

Consider two systems: $\dot{\mathbf{x}} = \widetilde{\Phi}(\mathbf{x})$ with $\mathbf{x} = (x_1, x_2, x_3) \in \Omega_1 \subset \mathbb{R}^3$ and $\dot{\mathbf{w}} = \Phi(\mathbf{w})$ with $\mathbf{w} = (w_1, w_2, w_3) \in \Omega_2 \subset \mathbb{R}^3$. Denote their solutions by $\widetilde{\phi}(t, \mathbf{x_0})$ and $\phi(t, \mathbf{w_0})$, with initial time $t_0 = 0$ and initial states $\mathbf{x_0}$ and $\mathbf{w_0}$.

Definition 2 (Ref. 18): The dynamics of the two different systems $\dot{\mathbf{w}} = \Phi(\mathbf{w})$ and $\dot{\mathbf{x}} = \widetilde{\Phi}(\mathbf{x})$ are called differentiable conjugate, if there exists a diffeomorphism $F : \mathbf{x} \in \mathbb{R}^3 \to \mathbf{w} \in \mathbb{R}^3$, namely, $\mathbf{w} = F(\mathbf{x})$, such that

$$\phi(t, F(\mathbf{x})) = F\big(\widetilde{\phi}(t, \mathbf{x})\big). \tag{2}$$

Taking derivative with respect to t on both sides of (2) yields

$$\begin{aligned} \frac{d}{dt}\phi\big(t, F(\mathbf{x})\big) &= \frac{d}{dt}F\big(\widetilde{\phi}(t, \mathbf{x})\big) \\ &= \Phi(\phi(t, F(\mathbf{x}))) = \mathrm{Jac}(F)\big(\widetilde{\phi}(t, \mathbf{x})\big) \cdot \widetilde{\Phi}\big(\widetilde{\phi}(t, \mathbf{x})\big). \end{aligned} \tag{3}$$

By letting $t = 0$ in (3), one has

$$\dot{\mathbf{x}} = \widetilde{\Phi}(\mathbf{x}) = (\mathrm{Jac}(F))^{-1}\Phi(F(\mathbf{x})). \tag{4}$$

For any $t > 0$ in (3), one can change the initial position $\mathbf{x}$, in other words, one can regard $\widetilde{\phi}(t, \mathbf{x})$ as the new initial state instead of $\mathbf{x}$. Thus, it can be seen that (3) is equivalent to (4). Hence, to verify (2), it suffices to show (4).

To this end, a simple argument is to combine the conjugacy and the periodicity of the system function, which can be used to verify the coexistence of infinitely many attractors in many systems, as further illustrated below.

Consider the Lorenz-type system introduced in Eq. (3.1) of Ref. 10,

$$\begin{cases} \dot{x}_1 = 10(\sin(x_2) - \sin(x_1)), \\ \dot{x}_2 = 28\sin(x_1) - 2R_0\sin(x_1)\sin(x_3) - \sin(x_2), \\ \dot{x}_3 = \frac{R_0}{2}\sin(x_1)\sin(x_2) - \frac{8}{3}\sin(x_3), \end{cases} \tag{5}$$

where R_0 is a positive constant parameter. Some numerical experiments provided in Ref. 10 illustrate the possible coexistence of infinitely many chaotic attractors all over the three-dimensional space for properly chosen values of R_0.

Consider the Rössler-type system introduced in Eq. (3.2) of Ref. 10,

$$\begin{cases} \dot{x}_1 = -\sin(x_2) - \sin(x_3), \\ \dot{x}_2 = \sin(x_1) + 0.2\sin(x_2), \\ \dot{x}_3 = \frac{0.2}{R_0} + \sin(x_3)(R_0\sin(x_1) - 5.7), \end{cases} \tag{6}$$

where R_0 is a positive constant parameter. Some numerical experiments provided in Ref. 10 illustrate the possible coexistence of infinitely many chaotic attractors all over the three-dimensional space for properly chosen values of R_0.

Theorem 3: *Assume that there is an attractor $\mathcal{A}$ in the system (5) or (6) for properly chosen values of R_0, and the basin of this attractor, denoted by U, is contained in a bounded set. Assume also that $(x_1^0, x_2^0, x_3^0) \in U$ is an initial state such that the orbit starting from this point will approach the attractor. Then, there is another attractor, for which the initial state can be taken as $(x_1^0 + 2k\pi, x_2^0 + 2m\pi, x_3^0 + 2r\pi)$, where k, m, and r are three integers, and k, m, and r are properly chosen values such that $(x_1^0 + 2k\pi, x_2^0 + 2m\pi, x_3^0 + 2r\pi) \notin U$. Consequently, there exist infinitely many attractors from the system (5) or (6), which can be found from the attractor $\mathcal{A}$.*

This theorem asserts that, under the above-mentioned parameter and initial conditions, if system (5) or (6) has one attractor then it will have infinitely many attractors.

Proof. In the system (5) or (6), the sinusoidal function has period 2π. The existence of other attractors can be verified via a simple conjugacy function.

Now, the argument is illustrated by using the system (5). The same argument can also be verified using the system (6).

Consider the system $\dot{\mathbf{x}} = \widetilde{\Phi}(\mathbf{x})$ represented by (5), namely,

$$\begin{cases} \dot{x}_1 = 10(\sin(x_2) - \sin(x_1)), \\ \dot{x}_2 = 28\sin(x_1) - 2R_0\sin(x_1)\sin(x_3) - \sin(x_2), \\ \dot{x}_3 = \frac{R_0}{2}\sin(x_1)\sin(x_2) - \frac{8}{3}\sin(x_3), \end{cases} \tag{7}$$

with initial states (x_1^0, x_2^0, x_3^0).

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Consider the system $\dot{\mathbf{w}} = \Phi(\mathbf{w})$ represented by

$$\begin{cases} \dot{w}_1 = 10(\sin(w_2) - \sin(w_1)), \\ \dot{w}_2 = 28\sin(w_1) - 2R_0\sin(w_1)\sin(w_3) - \sin(w_2), \\ \dot{w}_3 = \dfrac{R_0}{2}\sin(w_1)\sin(w_2) - \dfrac{8}{3}\sin(w_3), \end{cases} \tag{8}$$

with initial states $\left(x_1^0 + 2k\pi, x_2^0 + 2m\pi, x_3^0 + 2r\pi\right)$.

Furthermore, consider the map $(w_1, w_2, w_3) = F(x_1, x_2, x_3) = (x_1 + 2k\pi, x_2 + 2m\pi, x_3 + 2r\pi)$. It is evident that this map F is invertible, and the inverse of this map is $F^{-1}(w_1, w_2, w_3) = (w_1 - 2k\pi, w_2 - 2m\pi, w_3 - 2r\pi)$. Moreover, the Jacobian of F is an identity matrix.

Since

$$\begin{cases} \dot{x}_1 = 10(\sin(x_2) - \sin(x_1)) \\ \quad = 10(\sin(x_2 + 2m\pi) - \sin(x_1 + 2k\pi)), \\ \dot{x}_2 = 28\sin(x_1) - 2R_0\sin(x_1)\sin(x_3) - \sin(x_2) \\ \quad = 28\sin(x_1 + 2k\pi) - 2R_0\sin(x_1 + 2k\pi)\sin(x_3 + 2r\pi) \\ \qquad - \sin(x_2 + 2m\pi), \\ \dot{x}_3 = \dfrac{R_0}{2}\sin(x_1)\sin(x_2) - \dfrac{8}{3}\sin(x_3) \\ \quad = \dfrac{R_0}{2}\sin(x_1 + 2k\pi)\sin(x_2 + 2m\pi) - \dfrac{8}{3}\sin(x_3 + 2r\pi), \end{cases}$$

one has (4), that is,

$$\dot{\mathbf{x}} = \widetilde{\Phi}(\mathbf{x}) = (\mathrm{Jac}(F))^{-1}\Phi(F(\mathbf{x})).$$

Therefore, there exist infinitely many attractors, since k, m, and r can be any integers. □

Remark 4: For many systems, numerical calculations can show the existence of one attractor or one periodic orbit under appropriate parameter and initial conditions. For specific examples, one is referred to Figs. 1–7 in Ref. 10. Thus, by the argument given in the proof of Theorem 3, one can assert the coexistence of infinitely many attractors or periodic orbits.

Consider the following systems, introduced, respectively, in Eq. (2) of Ref. 14:

$$\begin{cases} \dot{x}_1 = \sin(ax_2) - b\sin(x_1), \\ \dot{x}_2 = \sin(ax_3) - b\sin(x_2), \\ \dot{x}_3 = \sin(ax_1) - b\sin(x_3), \end{cases} \tag{9}$$

and in Eq. (4) of Ref. 14

$$\begin{cases} \dot{x}_1 = \sin(ax_2) - b\tan(x_1), \\ \dot{x}_2 = \sin(ax_3) - b\tan(x_2), \\ \dot{x}_3 = \sin(ax_1) - b\tan(x_3), \end{cases} \tag{10}$$

where a and b are real parameters.

Theorem 5: *Assume that the system (9) or (10) has an attractor $\mathcal{A}$, with parameter a being a rational number. Denote the basin of this attractor by U, which is assumed to be contained in a bounded set. Assume also that $(x_1^0, x_2^0, x_3^0) \in U$ is an initial state such that the orbit starting from this point approaches the attractor. Furthermore, let $a = \pm\frac{k}{m}$, where k and m are two positive integers without common divisor. Then, there exists another attractor, with initial states $\left(x_1^0 + 2ms_1\pi, x_2^0 + 2ms_2\pi, x_3^0 + 2ms_3\pi\right)$, where s_1, s_2, and s_3 can be any integers, which are properly chosen such that $\left(x_1^0 + 2ms_1\pi, x_2^0 + 2ms_2\pi, x_3^0 + 2ms_3\pi\right) \notin U$. Consequently, there exist infinitely many attractors in the system (9) or (10), which can be found from attractor $\mathcal{A}$.*

Proof. This can be verified by similar arguments to the proof of Theorem 3. □

Consider the following system, introduced in Eq. (3) of Ref. 19:

$$\begin{cases} \dot{x}_1 = ax_2^2 - 1, \\ \dot{x}_2 = bx_3, \\ \dot{x}_3 = 1.1\tan\left(\dfrac{5\pi x_1}{22} + \dfrac{3\pi}{44}\right) - x_2 - x_3, \end{cases} \tag{11}$$

where a and b are real parameters.

Theorem 6: *Assume that the system (11) has an attractor $\mathcal{A}$. Denote the basin of this attractor by U, which is assumed to be contained in a bounded set. Assume also that $(x_1^0, x_2^0, x_3^0) \in U$ is an initial state such that the orbit starting from this point approaches the attractor. Then, there exists another attractor, with initial states $\left(x_1^0 + \frac{22k}{5}, x_2^0, x_3^0\right)$, where k can be any integer, which are properly chosen such that $\left(x_1^0 + \frac{22k}{5}, x_2^0, x_3^0\right) \notin U$. Consequently, there exist infinitely many attractors in the system (11), which can be found from attractor $\mathcal{A}$.*

Proof. This can be verified by similar arguments to the proof of Theorem 3. □

Remark 7: In Ref. 20, similar argument was used to study some two-dimensional offset-boostable chaotic systems with periodic trigonometric components, which were constructed based on standard jerk flows. This note indicates that this argument can be applied to more general systems.

III. DISCRETE-TIME SYSTEMS

In this subsection, discrete-time systems with infinitely many attractors are studied.

For simplicity, only planar systems are discussed, and only sinusoidal function is selected as an example of the periodic functions. The argument derived here can be easily extended to dynamical systems in $\mathbb{R}^n$ with some other types of periodic functions.

Definition 8 (Ref. 18): Let $F: \mathbf{x} \in \Omega_1 \subset \mathbb{R}^2 \to \mathbf{w} \in \Omega_2 \subset \mathbb{R}^2$ be a function. The function F is said to be a homeomorphism if F is continuous, one-to-one, and surjective, and its inverse is also continuous, one-to-one, and surjective.

Consider two discrete systems: $\overline{\mathbf{x}} = \widetilde{\Phi}(\mathbf{x})$ with $\mathbf{x} = (x_1, x_2) \in \Omega_1 \subset \mathbb{R}^2$, and $\overline{\mathbf{w}} = \Phi(\mathbf{w})$ with $\mathbf{w} = (w_1, w_2) \in \Omega_2 \subset \mathbb{R}^2$.

Definition 9 (Ref. 18): The dynamics of the two different discrete systems $\overline{\mathbf{w}} = \Phi(\mathbf{w})$ and $\overline{\mathbf{x}} = \widetilde{\Phi}(\mathbf{x})$ are called topological conjugate, if there exists a homeomorphism $F: \mathbf{x} \in \mathbb{R}^2 \to \mathbf{w} \in \mathbb{R}^2$, namely, $\mathbf{w} = F(\mathbf{x})$, such that

$$\Phi \circ F(\mathbf{x}) = F \circ \widetilde{\Phi}(\mathbf{x}).$$

The Harper's chaotic map was first introduced as a model of electron movement on the plane $\mathbb{R}^2$ under the influence of the magnetic field.[21] It is described by[21]

$$\begin{cases} x_{n+1} = x_n + K\sin(y_n), \\ y_{n+1} = y_n - L\sin(x_n + K\sin(y_n)), \end{cases} \tag{12}$$

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where K and L are two real parameters.

Theorem 10: *The system (12) has periodical dynamics. Specifically, consider the subsets $U \subset \mathbb{R}^2$ and $U' = U + \{2k\pi\} \times \{2m\pi\}$, where k and m are any integers, where U′ can be regarded as a parallel movement of U. The dynamics of the Harper's map on U and U′ are conjugate. Specifically, let $(x_0, y_0) \in U$ be an initial point such that the orbit starting from this point stays in U, the dynamics between the orbit with the initial point (x_0, y_0) and the orbit with the initial point $(x_0 + 2k\pi, y_0 + 2m\pi)$ are equivalent under a conjugacy function.*

Remark 11: In Theorem 10, if the orbit with the initial point (x_0, y_0) is periodic/quasi-periodic/chaotic, then the orbit with the initial point $(x_0 + 2k\pi, y_0 + 2m\pi)$ is periodic/quasi-periodic/chaotic.

Proof. Consider the system $\widetilde{\Phi}(x, y)$ represented by

$$\begin{cases} x_{n+1} = x_n + K\sin(y_n), \\ y_{n+1} = y_n - L\sin(x_n + K\sin(y_n)), \end{cases}$$

where the initial point is (x_0, y_0).

Consider the system $\Phi(u, v)$ represented by

$$\begin{cases} u_{n+1} = u_n + K\sin(v_n), \\ v_{n+1} = v_n - L\sin(u_n + K\sin(v_n)), \end{cases}$$

where the initial point is $(x_0 + 2k\pi, y_0 + 2m\pi)$.

Now, consider the map $(u, v) = h(x, y) = (x + 2k\pi, y + 2m\pi)$. It is evident that this map h is invertible, and its inverse is $h^{-1}(u, v) = (u - 2k\pi, v - 2m\pi)$. So, one has

$$\begin{cases} x_{n+1} = x_n + K\sin(y_n) = [x_n + 2k\pi + K\sin(y_n + 2m\pi)] - 2k\pi, \\ y_{n+1} = y_n - L\sin(x_n + K\sin(y_n)) \\ \qquad = [y_n + 2m\pi - L\sin(x_n + 2k\pi + K\sin(y_n + 2m\pi))] - 2m\pi, \end{cases}$$

that is,

$$\widetilde{\Phi}(x, y) = h^{-1} \circ \Phi \circ h(x, y).$$

Therefore, the dynamics of $\widetilde{\Phi}$ and of Φ are conjugate. □

Remark 12: Theorem 10 explains similar dynamics as in the stochastic web.[15]

Consider the following system, introduced in Eq. (7) of Ref. 22:

$$\begin{cases} x_{n+1} = x_n + 1 - a[F(x_n)]^2 + G(y_n) - F(x_n), \\ y_{n+1} = y_n + bF(x_n) - G(y_n), \end{cases} \tag{13}$$

where $F(\cdot)$ and $G(\cdot)$ are two continuous functions, and $G(\cdot)$ is periodic with period T.

Theorem 13: *Assume that the system (13) has an attractor, and the basin of this attractor, denoted by U, is contained in a bounded set. Assume also that $(x_0, y_0) \in U$ is an initial point such that the orbit starting from this point approaches the attractor. Then, there is another attractor with initial point $(x_0, y_0 + kT)$, where k is an integer such that $(x_0, y_0 + kT) \notin U$. Consequently, there exist infinitely many attractors in the system.*

Proof. Consider the system $\widetilde{\Phi}(x, y)$ represented by

$$\begin{cases} x_{n+1} = x_n + 1 - a[F(x_n)]^2 + G(y_n) - F(x_n), \\ y_{n+1} = y_n + bF(x_n) - G(y_n), \end{cases}$$

where the initial point is (x_0, y_0).

Consider the system $\Phi(u, v)$ represented by

$$\begin{cases} u_{n+1} = u_n + 1 - a[F(u_n)]^2 + G(v_n) - F(u_n), \\ v_{n+1} = v_n + bF(u_n) - G(v_n), \end{cases}$$

where the initial point is $(x_0, y_0 + kT)$.

Now, consider the map $(u, v) = h(x, y) = (x, y + kT)$. It is evident that this map h is invertible, and its inverse is $h^{-1}(u, v) = (u, v - kT)$. So, one has

$$\begin{cases} x_{n+1} = x_n + 1 - a[F(x_n)]^2 + G(y_n) - F(x_n) \\ \qquad = x_n + 1 - a[F(x_n)]^2 + G(y_n + kT) - F(x_n), \\ y_{n+1} = y_n + bF(x_n) - G(y_n) \\ \qquad = [y_n + kT + bF(x_n) - G(y_n + kT)] - kT, \end{cases}$$

that is,

$$\widetilde{\Phi}(x, y) = h^{-1} \circ \Phi \circ h(x, y).$$

Therefore, the dynamics of $\widetilde{\Phi}$ and of Φ are conjugate. □

Next, consider the following systems introduced in Ref. 16:

$$\begin{cases} x_{n+1} = k\sin(\pi y_n)x_n, \\ y_{n+1} = y_n + x_n \end{cases} \tag{14}$$

and

$$\begin{cases} x_{n+1} = k\left(e^{-\cos(\pi y_n)} - 1\right)x_n, \\ y_{n+1} = y_n + x_n, \end{cases} \tag{15}$$

where k is a real parameter.

Theorem 14: *Assume that the system (14) or (15) has an attractor, and the basin of this attractor, denoted by U, is contained in a bounded set. Assume also that $(x_0, y_0) \in U$ is an initial point such that the orbit starting from this point approaches the attractor. Then, there is another attractor with initial point $(x_0 + 2k\pi, y_0 + 2m\pi)$, where k and m are two integers such that $(x_0 + 2k\pi, y_0 + 2m\pi) \notin U$. Consequently, there exist infinitely many attractors in the system.*

Proof. This can be verified by similar arguments to the proof of Theorem 10. □

Remark 15: For a special kind of discrete-time systems, similar arguments were used for proving Theorem 11 in Ref. 17. Note, however, that there are discrete-time systems with periodic functions that do not have infinitely many attractors.[23,24]

IV. CONCLUSION

In this note, a method for verifying the coexistence of infinitely many attractors is developed. For a real system, this method can be applied to establish the conjugacy relationship useful for proving the coexistence of infinitely many attractors; in other words, not all the (continuous and discrete) systems with periodic components have the coexistence of infinitely many attractors, but only some specific systems do.

Except for those systems with periodic components, there also exist other methods for verifying the coexistence of infinitely many attractors, including some coupled chaotic systems without periodic components.[25–28] For the method developed in these systems, which do not have periodic components, more detailed analysis will be carried out in the near future.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Xu Zhang: Formal analysis (lead); Writing – original draft (equal). **Mingtao Chen:** Writing – original draft (equal). **Ran Zhang:** Writing – original draft (equal). **Guanrong Chen:** Writing – review & editing (lead).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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